1. (24%) Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size n=3 from the exponential distribution with p.d.f. $f(x) = e^{-x}$, $0 < x < \infty$.

(a) Define the order statistics. (6%)

(b) Find the p.d.f. of the sample median Y_2. (6%)

(c) Compute the probability that Y_3 is less than two. (6%)

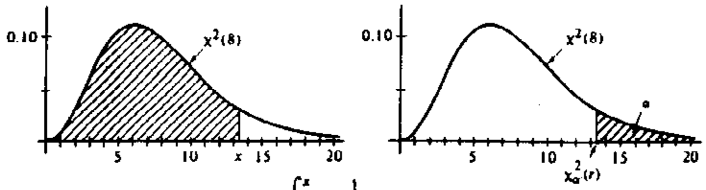
(d) Determine $P(1 < Y_1)$. (6%)

2. (26%) Analysis of the goals scored per match by a certain football team gave the following results:

No. goals per match (x)	0	1	2	3	4	5	6	7
No. of matches (f)	14	18	29	18	10	7	3	1

Calculate the mean of the above distribution and the frequencies (each correct to 1 decimal place) associated with a Poisson distribution having the same mean. Perform a χ^2 goodness of fit test to determine whether or not the above distribution can be reasonably modeled by this Poisson distribution.

Table V The Chi-Square Distribution



$$P(X \le x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} w^{r/2-1} e^{-w/2}\, dw$$

	$P(X \le x)$							
	0.010	0.025	0.050	0.100	0.900	0.950	0.975	0.990
r	$\chi^2_{0.99}(r)$	$\chi^2_{0.975}(r)$	$\chi^2_{0.95}(r)$	$\chi^2_{0.90}(r)$	$\chi^2_{0.10}(r)$	$\chi^2_{0.05}(r)$	$\chi^2_{0.025}(r)$	$\chi^2_{0.01}(r)$
1	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635
2	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210
3	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.34
4	0.297	0.484	0.711	1.064	7.779	9.488	11.14	13.28
5	0.554	0.831	1.145	1.610	9.236	11.07	12.83	15.09
6	0.872	1.237	1.635	2.204	10.64	12.59	14.45	16.81
7	1.239	1.690	2.167	2.833	12.02	14.07	16.01	18.48
8	1.646	2.180	2.733	3.490	13.36	15.51	17.54	20.09
9	2.088	2.700	3.325	4.168	14.68	16.92	19.02	21.67
10	2.558	3.247	3.940	4.865	15.99	18.31	20.48	23.21
11	3.053	3.816	4.575	5.578	17.28	19.68	21.92	24.72
12	3.571	4.404	5.226	6.304	18.55	21.03	23.34	26.22
13	4.107	5.009	5.892	7.042	19.81	22.36	24.74	27.69
14	4.660	5.629	6.571	7.790	21.06	23.68	26.12	29.14
15	5.229	6.262	7.261	8.547	22.31	25.00	27.49	30.58
16	5.812	6.908	7.962	9.312	23.54	26.30	28.84	32.00
17	6.408	7.564	8.672	10.08	24.77	27.59	30.19	33.41
18	7.015	8.231	9.390	10.86	25.99	28.87	31.53	34.80
19	7.633	8.907	10.12	11.65	27.20	30.14	32.85	36.19
20	8.260	9.591	10.85	12.44	28.41	31.41	34.17	37.57
21	8.897	10.28	11.59	13.24	29.62	32.67	35.48	38.93
22	9.542	10.98	12.34	14.04	30.81	33.92	36.78	40.29
23	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64
24	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98
25	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31
26	12.20	13.84	15.38	17.29	35.56	38.88	41.92	45.64
27	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96
28	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28
29	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59
30	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89
40	22.16	24.43	26.51	29.05	51.80	55.76	59.34	63.69
50	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15
60	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38
70	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.4
80	53.34	57.15	60.39	64.28	96.58	101.9	106.6	112.3

This table is abridged and adapted from Table III in *Biometrika Tables for Statisticians*, edited by E. S. Pearson and H. O. Hartley. It is published here with the kind permission of the *Biometrika* Trustees.

3.(30%) Imagine that you have recently been put in charge of the retail brokerage side of a large firm and you have a list of customers who have ever done business with the firm, as well as information on the frequency and timing of each customer's transactions. Now you are interested in the question that with a given purchase pattern what levels of transactions should be expected next period (of unit length) by those on the list, both individually and collectively.

Assume the interpurchase time of an individual follows an exponential distribution with mean $\frac{1}{\lambda}$. We also assume that different customers have different λ's (that is, λ is implicitly subscripted by a customer index, i). If we further suppose that the parameter λ, governing an individual customer's purchasing pattern, is Erlang distributed, i.e., λ follows the Erlang(γ, α) distribution, please aggregate (or "add up") across all customers to find the average probability of purchase for a random population member in a period (of unit length). Suppose that we observe an individual who makes c purchases in Period 1 (of unit length). How many purchases do we expect this same person to make in a nonoverlapping Period 2 (also of unit length)?

The density function of the Erlang distribution is

$$f(x|\gamma,\alpha) = \begin{cases} \frac{\alpha^\gamma}{(\gamma-1)!}x^{\gamma-1}e^{-\alpha x}, & \text{for } x > 0; \\ 0, & \text{for } x \leq 0, \end{cases}$$

where γ is an integer.

4.(20%) Suppose that a fair coin is tossed repeatedly until exactly 8 heads and at least one tail have been obtained. Determine the expected number of tosses that will be required.