國立臺灣科技大學
八十八學年度碩士班招生考試試題

系所別：電機工程系碩士班　　組別：乙組　　科目：工程數學

1. Let $\mathbf{v}_1 = (4, 6, 7)^T$, $\mathbf{v}_2 = (0, 1, 1)^T$, $\mathbf{v}_3 = (0, 1, 2)^T$, and let $\mathbf{u}_1 = (1, 1, 1)^T$, $\mathbf{u}_2 = (1, 2, 2)^T$, $\mathbf{u}_3 = (2, 3, 4)^T$.

   (a) Find the transition matrix from $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. (5%)

   (b) If $\mathbf{x} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$, determine the coordinates of $\mathbf{x}$ with respect to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. (5%)

2. Given

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$$

   (a) Determine the rank of A and the nullity of A. (5%)

   (b) Find an orthonormal basis for the nullspace of A. (5%)

3. Let an $m \times n$ matrix A representing the linear transformation from R^n to R^m, and let $\mathbf{b} \in R^m$. The vector $\mathbf{p}$ is said to be the projection of $\mathbf{b}$ onto $R(A)$, where $R(A)$ denote the range of A.

   (a) If rank $A = n$, find the projection vector $\mathbf{p}$. (5%)

   (b) If rank $A = k$ ($k < n$), and let the $R(A)$ spanned by the orthonormal basis $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k]$. Find the projection vector $\mathbf{p}$. (5%)

4. Evaluate $\oint_C \frac{\sin(z)\,dz}{z^2(z^2+4)}$, where C encloses 0 and $2i$. (10%)

5. Assume that $P(z)$ is a polynomial of degree ≥ 2, with zeros $w_1, w_2, \ldots, w_k$, with no w_j an integer. Using the residue theorem, however, we can obtain the formula

$$\sum_{n=-\infty}^{\infty} \frac{1}{P(n)} = -\pi \sum_{j=1}^{k} \operatorname*{Res}_{w_j}\left(\frac{\cot(\pi z)}{P(z)}\right).$$

   Using above formula, sum the series $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$. (10%)

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6. Let y_1 and y_2 be linear independent solutions of $x^2y''+2xy'+(x-2)y=0$. Then the Wronskian of y_1 and y_2 can be defined as $W(x)=y_1y_2'-y_1'y_2$. Given $y_1(1)=0$, $y_1'(1)=1$, $y_2(1)=2$, and $y_2'(1)=3$, find $W(x)$. (10%)

7. Given $\mathcal{L}(f(t))=F(s)$, use the convolution theorem to show that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}F(s)\right)=\int_0^t\int_0^\tau f(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\tau,$$

where $\mathcal{L}$ is the *Laplace transform* and $\mathcal{L}^{-1}$ is the *inverse Laplace Transform*. (10%)

8. Let $P_n(x)$ and $P_m(x)$ be two Legendre polynomials with $n\neq m$. Prove that

$$\int_{-1}^{1}P_n(x)P_m(x)\,\mathrm{d}x=0\ . \text{ (10\%)}$$

Hint: Legendre polynomials are the solutions of $(1-x^2)y''-2xy'+\lambda y=0$ for different λ with finite boundary at x=1 and x=-1.

9. An urn contains three black balls, four red balls and five white balls. Two balls are selected at random from the urn without replacement and the sequence of colors is noted.
   a. Find the probability that both balls are white. (3%)
   b. Find the probability that the second ball is red. (3%)
   c. If the experiment is with replacement, repeat the above two problems. (4%)

10. The *exponential random variable*, X, with a parameter λ has the *cumulative distribution function* (CDF) as

$$F_X(x)=\begin{cases}0 & x<0\\ 1-e^{-\lambda x} & x\geq 0\end{cases}.$$

Show that the exponential random satisfies the memoryless property; i.e. $P[\,X>t+h\mid X>t\,]=P[\,X>h\,]$. (10%)