國立臺灣科技大學
九十三學年度碩士班考試試題

系所組別：資訊工程系
科　　目：線性代數與離散數學

總分 100 分

1. (10%) Let $A=\begin{bmatrix}0.4 & -0.3\\ 0.4 & 1.2\end{bmatrix}$. Find $\lim_{k\to\infty} A^k$ where $A^k=\overbrace{AA\cdots A}^{k}$.

2. (10%) Find the projection of the vector $\mathbf{x}=\begin{bmatrix}2\\1\\4\end{bmatrix}$ onto the subspace spanned by $\beta=\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}\right\}$.

3. (10%) Let $C[-\pi,\pi]$ be the vector space of all continuous functions whose domain is the closed interval $[-\pi,\pi]$. We define the inner product by $\langle f,g\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$ where $f,g\in C[-\pi,\pi]$. Show that the functions $\sin x, \sin 2x, \ldots, \sin mx, \cos x, \cos 2x\ldots, \cos nx$ are *orthogonal* to one another where m,n are positive integers.

4. Label the following statements as "True" or "False". You have to prove it if it was "True" otherwise provide a counterexample.
Warning: *You will not get any point without the justification of your answer.*
   a. (5%) Let A and B be two nonsingular matrices. Then $A+B$ is an invertible matrix.

   b. (5%) Let the vector space V can be decomposed as the sum of two subspaces V_1 and V_2 denoted by $V=V_1+V_2$. Let W be a subspace of V. Then $W=(W\cap V_1)+(W\cap V_2)$.

   c. (5%) Let A be a *positive semi-definite* matrix in $R^{n\times n}$ and I_n be the *identity* matrix. Then $A+I_n$ is a *positive definite* matrix.

   d. (5%) Let A be a matrix in $R^{n\times n}$ and $\operatorname{rank}(A)\leqslant n-1$. Then the linear system $Ax=b$ always has a solution for all $b\in R^n$.

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5. (10%) Find all solutions of the congruence $x^2 \equiv 16 \pmod{105}$.

6. (8%) Show that if G is a bipartite simple graph with v vertices and e edges, then $e \le v^2/4$.

7. (10%) Professor Ruth has five graders to correct programs in her courses in Java, C++, SQL, Perl, and VHDL. Graders Jeanne and Charles both dislike SQL, Sandra wants to avoid C++ and VHDL. Paul detests Java and C++, and Todd refuses to work in SQL and Perl. In how many ways can Professor Ruth assign each grader to correct programs in one language, cover all five languages, and keep everyone content?

8. (a)(4%) Let $a, b, c \in \mathrm{Z}^+$, with $b \ge 2$. Let $f: \mathrm{Z}^+ \to \mathrm{R}^+ \cup \{0\}$ be a monotone increasing function, where

$$f(1) \le c \text{ and } f(n) \le af(n/b) + cn, \quad \text{for } n = b^k, \ k \in \mathrm{Z}^+.$$

Show that for all $n = 1, b, b^2, b^3, \ldots,$

$$f(n) \le cn \sum_{i=0}^{k} (a/b)^i.$$

(b)(4%) Use the result of part (a) to show that $f(n) = O(n \log n)$, where $a = b$. (The base for the log function here is any real number greater than 1.)

(c)(4%) When $a \ne b$, show that part (a) implies that

$$f(n) \le \left(\frac{c}{a-b}\right)\left(a^{k+1} - b^{k+1}\right).$$

9. Table 1 defines the next state function v and the output function ω for a finite state machine $M = (S, I, O, v, \omega)$, where $S = \{s_0, s_1\}$ and $I = O = \{0, 1\}$.

(a)(4%) Draw the state diagram for M.

(b)(3%) Determine the output for the following input sequences, starting at s_0 in each case: (i) $x = 111$; (ii) $x = 1010$; (iii) $x = 00011$.

(c)(3%) Describe in words what machine M does.

Table 1

	v		ω	
	0	1	0	1
s_0	s_0	s_1	0	0
s_1	s_0	s_1	1	1