國立台灣科技大學九十五學年度碩士班招生試題

系所組別：電機工程系碩士班丙一組、丙二組

科　　目：線性代數與機率

總分：100 分

1. (8%) Let $\mathbf{x}_1,\ldots,\mathbf{x}_k$ be linearly independent vectors in R^n and let A be a nonsingular $n\times n$ matrix. Now define $\mathbf{y}_i = A^T A\mathbf{x}_i$ for $i=1,\ldots,k$. Show that $\mathbf{y}_1,\ldots,\mathbf{y}_k$ are linearly independent. Note: A^T is the transpose of A.

2. (20%) $V=R^{2\times2}$, T: $V\to V$ is a linear transformation given by

$$T(A)=\frac{3A^T-A}{2}\quad \forall A\in V.$$

(a) (4%) Show that T is a linear operator.

(b) (5%) Find $[T]_\beta$, where β is a basis of V:

$$\beta=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&2\\-1&0\end{bmatrix}\begin{bmatrix}0&-1\\2&0\end{bmatrix}\begin{bmatrix}0&0\\0&2\end{bmatrix}\right\}.$$

(c) (5%) Find the eigenvalues and corresponding eigenvectors of $[T]_\beta$.

(d) (6%) Find the normalized QR-decomposition of $[T]_\beta$.

3 (12%) Let $P=A(A^TA)^{-1}A^T$, where A is a $m\times n$ matrix of rank n.

(a) (4%) Show that $P^k=P$ for $k=1,2,\ldots$.

(b) (4%) Show that P is symmetric.

(c) (4%) Let $A=\begin{bmatrix}0&1\\1&-1\\0&0\end{bmatrix}$, what is the projection of (1,1,1) onto the plane spanned by the column vectors of A?

4. (10%) Find a new coordinate x',y' so that the quadratic form $3x^2+2\sqrt{2}xy+2y^2$ can be written as $\lambda_1(x')^2+\lambda_2(y')^2$. What is the angle between the two coordinate systems?

123

國立台灣科技大學九十五學年度碩士班招生試題

系所組別： 電機工程系碩士班丙一組、丙二組

科　　目： 線性代數與機率

5. (12%) Suppose that a student passes the midterm exam with probability 0.6, passes the final exam with probability 0.7, and fails both exams with probability 0.25.
   (a) (6%) Find the probability that he passes the midterm exam or the final exam but not both.
   (b) (6%) Find the conditional probability that he passes the final exam, given that he fails the midterm exam.

6. (24%) The two random variables X and Y are independent with probability density functions $f_X(x)=\begin{cases} e^{-x}, & x\ge 0 \\ 0, & x<0 \end{cases}$ and $f_Y(y)=\begin{cases} e^{-y}, & y\ge 0 \\ 0, & y<0 \end{cases}$.

   Let $Z=X+Y$, $W=\dfrac{X}{X+Y}$ and $U=\dfrac{X}{Y}$.
   (a) (6%) Find the probability density function of U.
   (b) (6%) Find the joint probability density function of Z and W.
   (c) (6%) Find the probability density function of W.
   (d) (6%) Find the variance of Z.

7. (14%) A random variable S is given by $S=\sum_{i=1}^{N} X_i$ where the random variables X_i are independent, identically distributed with the probability density function $f_X(x)=\dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$.
   (a) (5%) Find the characteristic function of S.
   (b) (9%) Now suppose that N is a random variable taking the values 1, 2, … and is independent of a sequence of X_i. Given the mean of N is 10 and the variance of N is 5, find the mean and variance of S.

124