國立台灣科技大學九十八學年度碩士班招生試題

系所組別： 資訊工程系碩士班

科　　目： 計算機數學

（總分爲 100 分）

1. (a) If 11 integers are selected from {1, 2, 3, …, 100}, prove that there are at least two, say x and y, such that $0 < \left|\sqrt{x} - \sqrt{y}\right| < 1$. **(4%)**

   (b) Write a statement that generalizes the result of part (a). **(3%)**

2. In how many ways can one arrange all of the letters in the word INFORMATION so that no pair of consecutive letters occurs more than once? **(8%)**

3. (a) Show that the recurrence relation $f(n)a_n = g(n)a_{n-1} + h(n)$, for $n \geq 1$, and with $a_0 = C$, can be reduced to a recurrence relation of the form $b_n = b_{n-1} + Q(n)h(n)$, where

   $$b_n = g(n+1)Q(n+1)\,a_n, \text{ with } Q(n) = \frac{f(1)f(2)\cdots f(n-1)}{g(1)g(2)\cdots g(n)}.$$ **(8%)**

   (b) Use part (a) to solve the original recurrence relation to obtain

   $$a_n = \frac{C + \sum_{i=1}^{n} Q(i)h(i)}{g(n+1)Q(n+1)}.$$ **(4%)**

4. Construct a nondeterministic finite-state automaton that recognizes the language generated by the regular grammar $G = (V, T, S, P)$, where $V = \{0, 1, S, A, B\}$, $T = \{0, 1\}$, S is the start symbol, and the set P of productions is

   $$S \to 1B,\ S \to 0,\ A \to 1A,\ A \to 0B,$$
   $$A \to 1,\ A \to 0,\ B \to 1.$$ **(8%)**

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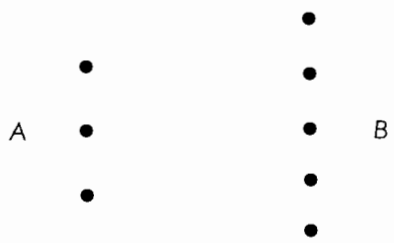
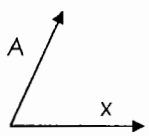
5. What is the maximum number of edges for a simple planar bipartite graph shown below? That is, no loops, multiple edges, crossing edges and no edges within groups A or B are allowed. You need to justify your answer by rigorous statements. (Hint: Try Euler's formula.)

(15%)



6. (a) Let us draw the vectors $x = [1, 0]$ and Ax on R^2 with the matrix $A = \begin{bmatrix} 1/4 & 3/4 \\ 1 & 1/2 \end{bmatrix}$.



Now let vector x go counter-clockwise for a cycle and come back to the original place. At the same time, the vector Ax will change accordingly. How many times vectors x and Ax will switch over each other? If your answer is greater than zero, please compute the ratio $|Ax|=|x|$ for each time vector x and vector Ax coincide with each other. Also the vector x should be mentioned for each of these coincidences. (Hint: x and vector Ax are in the same direction when they coincide with each other.) **(8%)**

(b) Same question again, but for a different matrix B where $B = \begin{bmatrix} 3/4 & 1/4 \\ -1/2 & 1 \end{bmatrix}$. **(7%)**

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7. (a) Let W be a finite-dimensional subspace of an inner product space V and u be a vector in V. Show that $\|u-\text{proj}_w u\| \leq \|u-w\|$ for every vector w in W. **(8%)**
   (b) Find the least squares straight line that fits to the following four points: (0,1), (1,3), (2,4), and (3,4). **(7%)**

8. Prove that if a square matrix A can be reduced to a row-echelon form U by Gaussian elimination without row interchanges, then A can be factored as $A=LU$, where L is a lower triangular matrix. **(10%)**

9. True or false? (A correct answer receives 2 points, while an incorrect answer receives -1 point and unanswered questions receive 0 point. The minimum total score of this problem is 0.) **(10%)**
   (a) Let V and W be finite-dimensional vector spaces and T be a linear transformation from V into W, then T maps linearly independent subsets of V into linearly independent subsets of W.
   (b) Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ, $[T]_\beta$ is similar to $[T]_\gamma$.
   (c) Every system of n linear equations in n unknowns can be solved by Cramer's rule.
   (d) The sum of two eigenvectors of a linear operator T is also an eigenvector of T.
   (e) $\text{rank}(AB)=\min\{\text{rank}(A), \text{rank}(B)\}$.